

Lost time: $S_n \subset V = \mathbb{C}^n$,

$W \subset V$ defined as

$$W = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : \sum_{j=1}^n a_j = 0 \right\}.$$

$S_n \subset W$.

Claim: The repn $S_n \subset W$ is irreducible

Pf: Suppose $w \in W$ is nonzero. Define $U = \text{Span}(g \cdot w : g \in G = S_n)$

Suffices to check: $U = W$.

with $w = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n$; not all λ_i 's are equal.

$\Rightarrow \exists i < j$ with $\lambda_i \neq \lambda_j$

Apply p with $p(i)=1, p(j)=2$.

$$p \cdot w = \lambda_i e_1 + \lambda_j e_2 + \mu_3 e_3 + \dots + \mu_n e_n$$

Apply (12) to the above

$$\Rightarrow (12) \cdot (p \cdot w) = \lambda_j e_1 + \lambda_i e_2 + \mu_3 e_3 + \dots + \mu_n e_n$$

$$\Rightarrow (12)(p)w - pw = (\lambda_j - \lambda_i)(e_1 - e_2) \in U$$

$$\Rightarrow e_1 - e_2 \in U.$$

$$\Rightarrow e_2 - e_3 \in U, e_3 - e_4 \in U, \dots, e_{n-1} - e_n \in U.$$

$$\text{Span}(e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n) = W \Rightarrow W \subset U \Rightarrow W = U$$

$$\text{Span}(e_1 - e_2, \dots, e_{n-1} - e_n) = W \Rightarrow W \subseteq U \Rightarrow W = U.$$

Proof of Maschke's Thm

Hermitian forms

$\mathbb{C}^n$, like $\mathbb{R}^n$, has an inner product $\langle, \rangle$

$$\left\langle \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2 + \dots + \bar{z}_n w_n.$$

A hermitian form on a $\mathbb{C}$ -space V , is $\langle, \rangle : V \times V \rightarrow \mathbb{C}$ satisfying

- (1) $\langle z + z', w \rangle = \langle z, w \rangle + \langle z', w \rangle$
- (2) $\langle z, w + w' \rangle = \langle z, w \rangle + \langle z, w' \rangle$
- (3) $\langle \lambda z, w \rangle = \bar{\lambda} \langle z, w \rangle, \quad \lambda \in \mathbb{C}$
- (4) $\langle z, \lambda w \rangle = \lambda \langle z, w \rangle$
- (5) $\langle w, z \rangle = \overline{\langle z, w \rangle}$

Note that $\langle, \rangle$ is a Hermitian form.

A Herm form is said to be positive definite if $\langle z, z \rangle \geq 0 \quad \forall z \in V$

$$\text{and } \langle z, z \rangle = 0 \Rightarrow z = 0.$$

Thm $G \leq V$ is a ^{G -stable} repn. Then $\exists$ a pos-def Herm form $\langle, \rangle$ on V that is G -inv., i.e.

$$\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle \quad \forall v_1, v_2 \in V.$$

Pf: Take an arbitrary pos-def Herm form on V call it $\langle, \rangle$

Pf Take an arbitrary pos-def Herm form on V , call it $\{, \}$.

$$\text{Define } \langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} \{gv, gw\}.$$

$v, w \in V$

One checks that $\langle, \rangle$ is

- pos-def
- Herm form
- G -inv.

Lemma: Suppose $W \subseteq V$ is a repn, $W \subseteq V$ a G -inv subspace, and $\langle, \rangle$ is a G -inv pos-def Herm form on V .

$$\text{Define } W^\perp = \{v \in V : \langle w, v \rangle = 0 \ \forall w \in W\}.$$

Then $V \cong W \oplus W^\perp$ and $W^\perp$ is G -inv.

PF: That $V = W \oplus W^\perp$ as vspacen is general lin alg fact about Herm forms. See book, § 8.3 / Thm 8.4.5.

$W^\perp$ is invt: Suppose $v \in W^\perp$. Claim: $gv \in W^\perp$. Indeed,

$$\langle w, gv \rangle = \langle \underbrace{g^{-1}w}_{{g^{-1}w \in W}}, \underbrace{g^{-1}gv}_{{\substack{\uparrow \\ \forall w \in W \\ \forall v \in W^\perp}}} \rangle = \langle g^{-1}w, v \rangle = 0$$

$\Rightarrow W^\perp$ is G invt.

Corollary: Maschke's Thm: Iterate the above procedure.

Characters

Suppose G a f.g. gp.

Q: Are there infinitely many irreducible reps of G .

Ans: No. There are finitely many.

Defⁿ Suppose $\rho: G \rightarrow GL_n(\mathbb{C})$ or $\rho: G \rightarrow GL(V)$ is a repn.

The character of ρ , χ_ρ , is the function $\chi: G \rightarrow \mathbb{C}$ defined as $\chi_\rho(g) = \text{tr}(\rho(g))$

Note: (1) Trace of lin trans. is indep of choice of basis

(2) If $\rho \sim \rho'$ are equivalent reps, then

$$\rho'(g) = T \rho(g) T^{-1} \quad \text{for some } T \in GL_n(\mathbb{C}).$$

$$\Rightarrow \chi_{\rho'}(g) = \text{tr}(\rho'(g)) = \text{tr}(T \rho(g) T^{-1}) = \text{tr}(\rho(g)) = \chi_\rho(g).$$

$\Rightarrow$ equivalent reps have the same character

(3) If $\rho = \rho_1 \oplus \rho_2$ then $\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2}$

(4) If $g_1 = x g_2 x^{-1}$ then

$$\begin{aligned} \chi_\rho(g_1) &= \text{tr}(\rho(g_1)) = \text{tr}(\rho(x) \rho(g_2) \rho(x)^{-1}) = \text{tr}(\rho(g_2)) \\ &= \chi_\rho(g_2) \end{aligned}$$

$\Rightarrow \chi_\rho$ is constant on conjugacy classes

(5) $\chi_\rho(1) = \dim V$ if $\rho: G \rightarrow GL(V)$

$$(5) \quad \chi_p(1) = \dim V \quad \text{if} \quad \rho: G \rightarrow GL(V).$$

Suppose χ, χ' are characters. Define

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi'(g).$$

Thm G a fin. gp.

① The irred chars of G are orthonormal for the above inner form

② There are finitely many irred reps of G up to equivalence.

The number of which is the number of conj. classes in G .

③ Let $\rho_1, \rho_2, \dots, \rho_r$ represent the isom classes of irred reps of G .

and let $\chi_1, \dots, \chi_r$ denote the assoc. characters, and $d_1, \dots, d_r$ the dims of $\rho_1, \dots, \rho_r$. Then

$$|G| = d_1^2 + d_2^2 + \dots + d_r^2 \quad \text{and each } d_i \mid |G|.$$

Corollary Notation as above. Suppose ρ is a repⁿ of G

w/ char χ . Define $n_i := \langle \chi, \chi_i \rangle$.

Then

① $\chi = n_1 \chi_1 + n_2 \chi_2 + \dots + n_r \chi_r$

② $\rho = n_1 \rho_1 \oplus n_2 \rho_2 \oplus \dots \oplus n_r \rho_r$

③ The reps ρ & ρ' of G are isom $\Leftrightarrow$ their chars are equal.

③ The reps $\rho + \rho'$ of G are isom $\Leftrightarrow$ their chars are equal.

Pf: $\rho = m_1 \rho_1 \oplus \dots \oplus m_r \rho_r$ for some integers m_j .

$$\Rightarrow \chi = m_1 \chi_1 + \dots + m_r \chi_r.$$

$$n_i = \langle \chi, \chi_i \rangle = \langle m_1 \chi_1 + \dots + m_r \chi_r, \chi_i \rangle = m_i \quad \text{b/c the } \chi_k \text{'s are orthonormal.}$$

$$\Rightarrow n_i = m_i \Rightarrow \text{Thm follows.}$$
